

*-RING ORDERINGS

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ABSTRACT. We examine a number of $*$ -ring orderings, generalizing classical properties of $*$ -positive elements to $*$ -accretives. We also examine $*$ -rings satisfying versions of Blackadar's property (SP), generalizing some basic properties of Rickart $*$ -rings to Blackadar $*$ -rings.

Motivation. Orderings on the positive elements have long been fundamental to operator algebra theory. More recently, the larger class of accretive elements has become important for generalizing C^* -algebra theory to Banach algebras (see [BR14], [BO14], [Ble15] and the references therein). Here we make some observations relating to order and orthogonality on the $*$ -accretives in a purely algebraic context. This demonstrates that certain basic properties only require a weak fragment of the C^* -algebra structure. It might also serve as a guide for properties to look for in more general Banach algebras.

A more traditional approach to $*$ -rings would be to focus on just the projections and assume they correspond to annihilators (see [Ber72]). However, this does not apply to many (e.g. infinite dimensional separable) C^* -algebras. This motivates us to examine weaker conditions which correspond to Blackadar's property (SP) in the C^* -algebra case. This allows us to generalize some of the classical Rickart $*$ -ring theory, as we demonstrate in §10 and §11.

Outline. In §1 we make some general definitions for binary relations.

In §2 we discuss a number of semigroup orderings.

In §3 we review proper $*$ -rings and define an equivalence relation from the skew-adjoints. A weaker preorder is then defined from the $*$ -accretives in §4. The only assumptions we require are (A) and (B) which say that A is a proper unital $*$ -ring for which this preorder is antisymmetric on the self-adjoints.

In §5 we introduce some other important subsets of A and discuss their interrelationships, closure properties and the order relations they define.

In §6 we generalize orthogonality properties of $*$ -positive elements to $*$ -accretives, e.g. showing orthogonality is symmetric on \mathfrak{c} and \mathfrak{cc} contains no non-zero nilpotents.

In §7 we show that the fixator relation \ll is auxiliary to various other order relations and discuss lattice properties and Riesz interpolation for \ll .

In §8 we characterize projections and their products, sums and differences.

In §9 we use the extra assumption (C) to generalize some C^* -algebra results on $*$ -positive decompositions, square-roots and products.

In §10 we examine the relationships between various kinds of Blackadar $*$ -rings.

In §11 we characterize projection supremums/infimums in \subseteq_{\perp} -Blackadar $*$ -rings.

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1. ORDERINGS

We define the usual *composition* of relations $\ll, \preceq \subseteq A \times A$ by

$$\ll \circ \preceq = \bigcup_{c \in A} \{(a, b) \in A \times A : (a, c) \in \ll \text{ and } (c, b) \in \preceq\},$$

i.e. $a \ll \circ \preceq b \Leftrightarrow \exists c \in A (a \ll c \preceq b)$ in standard infix notation.

Generalizing the definition of auxiliaryity in [GHK⁺03] Definition I-1.11, we say

$$\begin{aligned} \ll \text{ is left auxiliary to } \preceq &\Leftrightarrow \ll \circ \preceq = \ll \subseteq \preceq. \\ \ll \text{ is right auxiliary to } \preceq &\Leftrightarrow \preceq \circ \ll = \ll \subseteq \preceq \\ \ll \text{ is auxiliary to } \preceq &\Leftrightarrow \ll \text{ is left and right auxiliary to } \preceq. \end{aligned}$$

Define $B \ll_F C \Leftrightarrow b \ll c$, for all $b \in B$ and $c \in C$, where $B, C \neq \emptyset$ are finite.

$$\begin{aligned} \ll \text{ is transitive} &\Leftrightarrow \ll \circ \ll \subseteq \ll. \\ \ll \text{ has interpolation} &\Leftrightarrow \ll \circ \ll \supseteq \ll. \\ \ll \text{ has Riesz interpolation} &\Leftrightarrow \ll_F \circ \ll_F \supseteq \ll_F. \end{aligned}$$

So if \ll is left (or right) auxiliary to \preceq then \ll is automatically transitive, for then $\ll \circ \ll \subseteq \ll \circ \preceq \subseteq \ll$. Also \ll is self-auxiliary iff \ll is transitive and has interpolation. Transitivity also means it suffices for Riesz interpolation to hold on pairs of elements in A . Further define the following standard terminology.

$$\begin{aligned} \ll \text{ is reflexive} &\Leftrightarrow = \subseteq \ll. \\ \ll \text{ is antisymmetric} &\Leftrightarrow = \supseteq \ll \cap \gg. \\ \ll \text{ is symmetric} &\Leftrightarrow \ll = \gg. \\ \ll \text{ is a preorder} &\Leftrightarrow \ll \text{ is transitive and reflexive.} \\ \ll \text{ is a partial order} &\Leftrightarrow \ll \text{ is an antisymmetric preorder.} \\ \ll \text{ is an equivalence relation} &\Leftrightarrow \ll \text{ is a symmetric preorder.} \end{aligned}$$

Primarily for use in §10 and §11, let $a \ll$ and $\ll a$ denote the subsets defined by

$$\begin{aligned} a \ll &= \{b \in A : a \ll b\}. \\ \ll a &= \{b \in A : b \ll a\}. \end{aligned}$$

For any relation $\preceq \subseteq \mathcal{P}(A) \times \mathcal{P}(A)$, where $\mathcal{P}(A) = \{B \subseteq A\}$, define

$$(1.1) \quad a \preceq^{\ll} b \Leftrightarrow (\ll a) \preceq (\ll b).$$

$$(1.2) \quad a \preceq_{\ll} b \Leftrightarrow (b \ll) \preceq (a \ll).$$

Note transitivity, reflexivity and symmetry hold for \preceq^{\ll} if they hold for \preceq . In particular, \subseteq^{\ll} and \subseteq_{\ll} are preorders while $=^{\ll}$ and $=_{\ll}$ are equivalence relations. Also, for any $\preceq \subseteq A \times A$,

$$(1.3) \quad \preceq \subseteq \subseteq^{\ll} \Leftrightarrow \ll \circ \preceq \subseteq \ll.$$

$$(1.4) \quad \preceq \subseteq \subseteq_{\ll} \Leftrightarrow \preceq \circ \ll \subseteq \ll.$$

Thus transitivity and reflexivity of \ll is characterized in terms of \subseteq^{\ll} (or \subseteq_{\ll}) by

$$\begin{aligned} \ll \subseteq \subseteq^{\ll} &\Leftrightarrow \ll \circ \ll \subseteq \ll. \\ \subseteq^{\ll} \subseteq \ll &\Leftrightarrow = \subseteq \ll. \end{aligned}$$

So \subseteq^{\ll} and \subseteq_{\ll} are the weakest relations having \ll as a left and right auxiliary respectively, as long as \ll is transitive. And they both coincide with \ll if \ll is a preorder. These constructions apply to non-transitive relations too, for example to the \ll -incompatibility relation \top defined by

$$a \top b \Leftrightarrow \forall c \in A (c \ll a, b \Rightarrow (c \ll) = A).$$

Separativity, as in [Kun80] Chapter 2 Exercise (15), is then naturally defined by

$$\ll \text{ is separative} \Leftrightarrow \ll = \subseteq_{\top}.$$

We also define *supremums* \bigvee and *infimums* \bigwedge of $B \subseteq A$ w.r.t. \ll by

$$(1.5) \quad a = \bigvee B \Leftrightarrow (a \ll) = \bigcap_{b \in B} (b \ll).$$

$$(1.6) \quad a = \bigwedge B \Leftrightarrow (\ll a) = \bigcap_{b \in B} (\ll b).$$

So when \ll is reflexive, supremums/infimums are upper/lower bounds. If \ll is also antisymmetric, then supremums/infimums are unique, when they exist. Also \ll -supremums/ \ll -infimums are precisely the \subseteq_{\ll} -supremums/ \subseteq^{\ll} -infimums so we could restrict to preorders here, as is often done in the literature. Also define

$$\begin{aligned} A \text{ is a } \ll\text{-semilattice} &\Leftrightarrow \bigvee F \text{ exists for all non-empty finite } F \subseteq A. \\ A \text{ is a } \ll\text{-lattice} &\Leftrightarrow A \text{ is a } \ll\text{-semilattice and } \gg\text{-semilattice.} \end{aligned}$$

For A to be a \ll -lattice it suffices that $a \vee b = \bigvee \{a, b\}$ and $a \wedge b = \bigwedge \{a, b\}$ exist/are defined, for all $a, b \in A$. This implies Riesz interpolation is equivalent to interpolation. For $B \subseteq A$, we also define

$$\begin{aligned} B \text{ is } \ll\text{-cofinal in } A &\Leftrightarrow B \cap (a \ll) \neq \emptyset \text{ whenever } (a \ll) \neq \emptyset. \\ B \text{ is } \ll\text{-coinital in } A &\Leftrightarrow B \cap (\ll a) \neq \emptyset \text{ whenever } (\ll a) \neq \emptyset. \end{aligned}$$

2. SEMIGROUPS

In a semigroup A we define the *Green*, *fixator* and *orthogonality* relations by

$$\begin{aligned} a \preceq b &\Leftrightarrow a \in Ab. \\ a \ll b &\Leftrightarrow a = ab. \\ a \perp b &\Leftrightarrow 0 = ab. \end{aligned}$$

So \perp requires a zero $0 \in A$ i.e. satisfying $0A = \{0\} = A0$. Also $(\preceq a) = Aa$ so

$$a \subseteq^{\preceq} b \Leftrightarrow Aa \subseteq Ab,$$

which provides an alternative description of \preceq when \preceq is reflexive, e.g. when A has a unit $1 \in A$, i.e. satisfying $1a = a = a1$, for all $a \in A$. In this case, the symmetrization $\mathcal{L} = \preceq \cap \succeq$ is well-known in semigroup theory as one of Green's relations (see [Law04] Chapter 10), while $\approx = \preceq \circ \preceq^{\text{op}}$, where $a \preceq^{\text{op}} b \Leftrightarrow a \in bA$, has been studied for C^* -algebra A in [Cun77]. Variants of \ll and \perp are also often considered in C^* -algebras – see [Bla13] II.3.1.13 and II.3.4.3. They also crop up naturally in lattice theory. Indeed, if \leq is a partial order making A a \geq -semilattice and we take \wedge as our semigroup operation then $\leq = \preceq = \ll$ and $\top = \perp$.

In general, we have the following relationships between \preceq , \ll and \perp .

$$\begin{aligned} (2.1) \quad & \preceq \circ \preceq \subseteq \preceq. \\ (2.2) \quad & \preceq \circ \ll \subseteq \ll \subseteq \preceq. \\ (2.3) \quad & \preceq \circ \perp \subseteq \perp. \end{aligned}$$

Proof.

- (2.1) If $a \preceq c \preceq b$ then $a = dc$ and $c = eb$ so $a = dc = deb$, i.e. $a \preceq b$.
- (2.2) If $a \preceq c \ll b$ then $a = dc$ and $c = cb$ so $a = dc = dcb = ab$, i.e. $a \ll b$.
- (2.3) If $a \preceq c \perp b$ then $a = dc$ and $0 = cb$ so $0 = d0 = dcb = ab$, i.e. $a \perp b$. \square

By (2.2), \ll is transitive. Applied to binary relations on A under composition, this shows that auxiliarity is itself a transitive relation. If \preceq is reflexive (e.g. if A is unital), (2.2) also shows that \preceq is right auxiliary to \ll .

We will also consider \preceq relative to various subsets B of A defined by

$$a \preceq b \iff a \in Bb.$$

If A is unital then we can characterize properties of \preceq by those of B as follows.

$$\begin{aligned} (\text{Transitivity}) \quad & BB \subseteq B \iff \preceq \circ \preceq \subseteq \preceq. \\ (\text{Reflexivity}) \quad & 1 \in B \iff = \subseteq \preceq. \\ (\text{Symmetry}) \quad & B^{-1} = B \iff \succeq = \preceq. \end{aligned}$$

Here B^{-1} denotes the inverses of invertible elements of B .

Proof.

- (Transitivity) If $\preceq \circ \preceq \subseteq \preceq$ and $a, b \in B$ then $ab \preceq b \preceq 1$ so $ab \preceq 1$, i.e. $ab \in B$.
- (Reflexivity) If $1 \in B$ then $a \in Ba$ so $a \preceq a$, for all $a \in A$. If $1 \preceq 1$ then $1 \in B1 = B$.
- (Symmetry) If $B^{-1} = B$ and $a \preceq b$ then $a \in Bb$ so $b \in B^{-1}a = Ba$, i.e. $b \preceq a$.
 If $\succeq = \preceq$ and $a \in B$ then $a \preceq 1$ so $1 \preceq a$ and hence $1 \in Ba$, i.e. a has a left inverse $a^{-1} \in B$. Likewise, a^{-1} has a left inverse $(a^{-1})^{-1} \in B$. But then $(a^{-1})^{-1} = (a^{-1})^{-1}1 = (a^{-1})^{-1}a^{-1}a = 1a = a$ so a is a left inverse of a^{-1} , i.e. a^{-1} is also a right inverse of a . \square

Often \preceq is considered when B is a subset of a group A . In this case, using additive notation, any subsemigroup $B = B + B$ containing 0 defines a preorder by

$$a \preceq b \iff b - a \in B,$$

which is an equivalence relation iff $B = -B$ is a subgroup, and a partial order iff

$$(\text{Antisymmetry}) \quad B \cap -B \subseteq \{0\} \iff \preceq \cap \succeq \subseteq =.$$

3. *-RINGS

Following [Ber72], we make the following standing assumption until §10.

(A) A is a proper unital¹ *-ring.

¹Unitality is required to define the unit ball \mathfrak{B} (see §5 below). However, any non-unital proper *-ring has a proper unitization (see [Ber72] §5 Definition 3), which could be used to generalize the theory. The only caveat is that different unitizations might yield different generalizations.

So the *adjoint* $*$ is a proper self-inverse morphism from A to A^{op} , i.e.

$$(3.1) \quad \begin{aligned} a^{**} &= a. \\ (ab)^* &= b^* a^*. \\ (a+b)^* &= a^* + b^*. \\ a^* a = 0 &\Rightarrow a = 0. \end{aligned}$$

The *self-adjoint*, *skew-adjoint* and *normal* elements are defined by

$$\begin{aligned} A_{\text{sa}} &= \{a \in A : a = a^*\}, \\ A_{\text{sk}} &= \{a \in A : a = -a^*\} \text{ and} \\ A_{\text{n}} &= \{a \in A : aa^* = a^*a\} \\ &\supseteq A_{\text{sa}} \cup A_{\text{sk}}. \end{aligned}$$

As $a^* + (-a)^* = (a - a)^* = 0^* = (0^*0)^* = 0^*0^{**} = 0^*0 = 0$, we have $(-a)^* = -a^*$. Thus $A_{\text{sk}} = -A_{\text{sk}} = A_{\text{sk}} + A_{\text{sk}}$ and we get an equivalence relation defined by

$$a \equiv b \Leftrightarrow a - b \in A_{\text{sk}}.$$

But $a - b \in A_{\text{sk}}$ means $a - b = -(a - b)^* = -a^* + b^*$ so

$$a \equiv b \Leftrightarrow a^* + a = b^* + b.$$

So \equiv is the equivalence relation coming from the $+$ -homomorphism $a \mapsto a^* + a$, which induces a $+$ -morphism from A/A_{sk} to A_{sa} . In particular, for all $a \in A$,

$$a \equiv a^*.$$

4. POSITIVITY

Define the **-squares*, **-sums*, **-positive* and **-accretive* elements by

$$\begin{aligned} |A|^2 &= \{a^*a : a \in A\}. \\ A_{\Sigma} &= \left\{ \sum_{k=1}^n a_k : a_1, \dots, a_n \in |A|^2 \right\}. \\ A_+ &= \{a \in A : na \in A_{\Sigma}, \text{ for some } n \in \mathbb{N}\}. \\ \mathfrak{r} &= \{a \in A : a + a^* \in A_+\}. \end{aligned}$$

The only other standing assumption we need until §9 is that A_{Σ} is salient, i.e.

$$(B) \quad A_{\Sigma} \cap -A_{\Sigma} = \{0\}.$$

This means that $(A, +)$ is torsion-free, for if $na = 0$ then $na^*a = 0$ and hence $a^*a = -(n-1)a^*a$, which means $a^*a = 0$, by (B), and hence $a = 0$, by (3.1).² This, in turn, means that A_+ is salient too, for if $a \in A_+ \cap -A_+$ then we have $m, n \in \mathbb{N}$ with $ma, -na \in A_{\Sigma}$ and hence $mna \in A_{\Sigma} \cap -A_{\Sigma} = \{0\}$ so $a = 0$. Thus

$$(4.1) \quad -A_+ \cap A_+ = \{0\}.$$

$$(4.2) \quad A_{\text{sk}} \cap A_{\text{sa}} = \{0\}.$$

It fact, for (4.2) it suffices that 2 is not a zero-divisor, i.e. $2(0 \neq) \subseteq (0 \neq)$.

²Conversely, (B) follows from (A), 2 is not a zero divisor in A , -1 has a square root in A' and every $a \in A_+$ has a square root in $A_{\text{sa}} \cap \{a\}''$, where $B' = \{a \in A : ab = ba\}$ – see [Ber72] §51.

For $B \subseteq A$ and $n \in \mathbb{N}$, define $\frac{1}{n}B = \{a \in A : na \in B\}$ so

$$\begin{aligned} B &= \frac{1}{n}nB. \\ \frac{1}{m}\frac{1}{n}B &= \frac{1}{mn}B. \\ B \subseteq \frac{1}{n}B &\Leftrightarrow nB \subseteq B. \end{aligned}$$

By definition, $A_+ = \bigcup_n \frac{1}{n}A_\Sigma$ so $\frac{1}{m}A_+ = \bigcup_n \frac{1}{mn}A_\Sigma \subseteq A_+$, for all $m \in \mathbb{N}$. If $ma, nb \in A_\Sigma$ then $mna, mnb \in A_\Sigma$ so $mn(a+b) \in A_\Sigma$, as $A_\Sigma = A_\Sigma + A_\Sigma$. Thus

$$\begin{aligned} \frac{1}{n}A_+ &= A_+ = A_+ + A_+ \quad \text{so} \\ \frac{1}{n}\mathfrak{r} &= \mathfrak{r} = \mathfrak{r} + \mathfrak{r}. \end{aligned}$$

Also $A_{\text{sa}} = \frac{1}{n}A_{\text{sa}}$, for if $na \in A_{\text{sa}}$ then $na = na^*$ so $n(a - a^*) = 0$ and $a = a^*$. Certainly $|A|^2 \subseteq A_{\text{sa}}$ so $A_\Sigma \subseteq A_{\text{sa}}$ and hence $A_+ \subseteq A_{\text{sa}}$ too, as $A_+ = \bigcup_n \frac{1}{n}A_\Sigma$. If $a \in A_{\text{sa}} \cap \mathfrak{r}$ then $2a = a + a^* \in A_+$ so $a \in \frac{1}{2}A_+ = A_+$. While if $a \in \mathfrak{r} \cap -\mathfrak{r}$ then $a + a^* \in A_+ \cap -A_+ = \{0\}$ and hence $a = -a^*$, i.e.

$$(4.3) \quad A_+ = \mathfrak{r} \cap A_{\text{sa}}.$$

$$(4.4) \quad A_{\text{sk}} = \mathfrak{r} \cap -\mathfrak{r}.$$

If $2 \in A^{-1}$ then $a = \frac{1}{2}(a + a^*) + \frac{1}{2}(a - a^*)$ so

$$(4.5) \quad 2 \in A^{-1} \quad \Rightarrow \quad \mathfrak{r} = A_+ + A_{\text{sk}}.$$

Also, for $a, b \in A$, $(ba)^*(ba) = a^*b^*ba$ so $a^*|A|^2a \subseteq |A|^2$, $a^*A_\Sigma a \subseteq A_\Sigma$ and $a^*A_+a \subseteq A_+$. Thus if $b \in \mathfrak{r}$ then $a^*ba + (a^*ba)^* = a^*(b + b^*)a \in a^*A_+a \subseteq A_+$, so we have

$$a^*\mathfrak{r}a \subseteq \mathfrak{r}.$$

As $A_+ + A_+ = A_+$ and $\mathfrak{r} = \mathfrak{r} + \mathfrak{r}$, we get a preorders \preceq^+ and $\preceq^{\mathfrak{r}}$ defined by

$$\begin{aligned} a \preceq^+ b &\Leftrightarrow b - a \in A_+. \\ a \preceq^{\mathfrak{r}} b &\Leftrightarrow b - a \in \mathfrak{r}. \end{aligned}$$

From now on \preceq is fixed as an abbreviation for $\preceq^{\mathfrak{r}}$.

By (4.1), \preceq^+ is a partial order which is traditionally only considered on A_{sa} . Thus \preceq provides a consistent extension to A . Indeed, (4.2) and (4.3) yield

$$\begin{aligned} \equiv &= \equiv \quad \text{on } A_{\text{sa}}. \\ \preceq &= \preceq^+ \quad \text{on } A_{\text{sa}}. \end{aligned}$$

Also, by (4.4), (4.5) and $a \equiv \frac{1}{2}(a^* + a) \preceq \frac{1}{2}(b^* + b) \equiv b$, when $2 \in A^{-1}$, we have

$$\equiv = \preceq \cap \succeq.$$

$$2 \in A^{-1} \quad \Rightarrow \quad \preceq = (\preceq^+ \circ \equiv) = (\equiv \circ \preceq^+) = (\equiv \circ \preceq_{\text{sa}}^+ \circ \equiv),$$

where \preceq_{sa}^+ denotes the restriction of \preceq^+ to A_{sa} . Also $a^*\mathfrak{r}a \subseteq \mathfrak{r} = \mathfrak{r}^* = \frac{1}{n}\mathfrak{r}$ means

$$nb \preceq nc \quad \Leftrightarrow \quad b^* \preceq c^* \quad \Leftrightarrow \quad b \preceq c \quad \Rightarrow \quad a^*ba \preceq a^*ca.$$

Lastly, denote the composition of $\preceq^{\mathfrak{r}}$ and $a \mapsto a^*a$ by \preceq^* so

$$a \preceq^* b \quad \Leftrightarrow \quad a^*a \preceq b^*b.$$

5. BALLS AND CONES

Define the balls \mathfrak{B} , $\frac{1}{2}\mathfrak{F}$ and \mathfrak{F} and the cone \mathfrak{c} by

$$\begin{aligned}\mathfrak{B} &= \{a \in A : a^*a \preceq 1\}, \\ \frac{1}{2}\mathfrak{F} &= \{a \in A : a^*a \preceq a\}, \\ \mathfrak{F} &= \{a \in A : a^*a \preceq 2a\}, \\ \mathfrak{c} &= \{a \in A : a^*a \preceq na, \text{ for some } n \in \mathbb{N}\}.\end{aligned}$$

Note $2a \in \mathfrak{F} \Leftrightarrow 4a^*a = (2a)^*(2a) \preceq 2(2a) = 4a \Leftrightarrow a^*a \preceq a \Leftrightarrow a \in \frac{1}{2}\mathfrak{F}$, so this is consistent with the fraction notation in §4. Further define operations

$$\begin{aligned}a^\perp &= 1 - a, \\ |a|^2 &= a^*a, \\ a \bullet b &= a + b - ab, \\ a * b &= a + b - 2ab.\end{aligned}$$

The associativity of \bullet and $*$ follows from the associativity of multiplication and

$$\begin{aligned}(a \bullet b)^\perp &= 1 - a - b + ab = a^\perp b^\perp, \\ 2(a * b) &= 2a + 2b - 4ab = (2a) \bullet (2b).\end{aligned}$$

In fact, this shows that $a \mapsto a^\perp$ is a $(*)$ -isomorphism from (A, \cdot) onto (A, \bullet) so (A, \bullet) is also a proper $*$ -semigroup. We also have the following.

$$\begin{aligned}\mathfrak{B}^\perp &= \mathfrak{F} \subseteq \mathfrak{c} \subseteq \mathfrak{r}, \\ \{0, 1\} &\subseteq |\mathfrak{B}|^2 \subseteq \frac{1}{2}\mathfrak{F} = (\frac{1}{2}\mathfrak{F})^\perp \subseteq \mathfrak{F} \cap \mathfrak{B}, \\ \mathfrak{B} &= \mathfrak{B}^*, \quad \frac{1}{2}\mathfrak{F} = \frac{1}{2}\mathfrak{F}^*, \quad \mathfrak{F} = \mathfrak{F}^*, \quad \mathfrak{c} = \mathfrak{c}^*, \\ \mathfrak{B}\mathfrak{B} &= \mathfrak{B}, \quad \mathfrak{F} \bullet \mathfrak{F} = \mathfrak{F}, \quad \frac{1}{2}\mathfrak{F} * \frac{1}{2}\mathfrak{F} = \frac{1}{2}\mathfrak{F}, \\ \mathfrak{c} + \mathfrak{c} &= \mathfrak{c}, \quad \mathfrak{c} \cap -\mathfrak{c} = \{0\}.\end{aligned}$$

Proof.

$$\begin{aligned}\mathfrak{B}^\perp &= \mathfrak{F} \text{ Note } a^{\perp*}a^\perp = 1 - a^* - a + a^*a \equiv 1 - 2a + a^*a \text{ so } a^*a \preceq 2a \Leftrightarrow a^{\perp*}a^\perp \preceq 1. \\ \mathfrak{F} &\subseteq \mathfrak{c} \subseteq \mathfrak{r} \text{ Note } 0 \preceq a^*a \preceq na \text{ yields } 0 \preceq a. \\ \{0, 1\} &\subseteq |\mathfrak{B}|^2 \text{ Note } 0 = 0^*0 \text{ and } 1 = 1^{**} = (1^*1)^* = 1^*1^{**} = 1^*1. \\ |\mathfrak{B}^*|^2 &\subseteq \frac{1}{2}\mathfrak{F} \text{ Note } aa^* \preceq 1 \text{ yields } (a^*a)^*a^*a = a^*(aa^*)a \preceq a^*a. \\ (\frac{1}{2}\mathfrak{F})^\perp &= \frac{1}{2}\mathfrak{F} \text{ Note } a \in \frac{1}{2}\mathfrak{F} \text{ means } a^{\perp*}a^\perp = 1 - a^* - a + a^*a \preceq a^{\perp*} \equiv a^\perp. \\ \frac{1}{2}\mathfrak{F} &\subseteq \mathfrak{F} \text{ Note } 0 \preceq a^*a \preceq a \text{ means } a^*a \preceq 2a^*a \preceq 2a. \\ \frac{1}{2}\mathfrak{F} &\subseteq \mathfrak{B} \text{ Note } \frac{1}{2}\mathfrak{F} = (\frac{1}{2}\mathfrak{F})^\perp \subseteq \mathfrak{F}^\perp = \mathfrak{B}. \\ \mathfrak{B} &= \mathfrak{B}^* \text{ If } a^*a \preceq 1 \text{ then } aa^*aa^* \preceq aa^* \text{ so } 0 \preceq (aa^*)^{\perp 2} = 1 - 2aa^* + aa^*aa^* \preceq (aa^*)^\perp. \\ \frac{1}{2}\mathfrak{F} &= \frac{1}{2}\mathfrak{F}^* \text{ and } \mathfrak{F} = \mathfrak{F}^* \text{ Note } \mathfrak{F} = \mathfrak{B}^\perp = \mathfrak{B}^{*\perp} = \mathfrak{F}^*. \\ \mathfrak{c} &= \mathfrak{c}^* \text{ As above for } \mathfrak{B} = \mathfrak{B}^*, \text{ we have } a^*a \preceq n \Leftrightarrow aa^* \preceq n. \text{ Now if } a \in \mathfrak{c} \text{ then } 0 \preceq a^*a \preceq na \preceq 2na. \text{ Then } (n-a)^*(n-a) \equiv n^2 - 2na + a^*a \preceq n^2 \text{ so } (n-a)(n-a)^* \preceq n^2 \text{ and hence } aa^* \preceq 2na^*. \text{ Thus } a \in \mathfrak{c}^*. \\ \mathfrak{B}\mathfrak{B} &= \mathfrak{B} \text{ If } a, b \in \mathfrak{B} \text{ then } a^*a \preceq 1 \text{ so } b^*a^*ab \preceq b^*b \preceq 1 \text{ hence } ab \in \mathfrak{B}. \\ \mathfrak{F} \bullet \mathfrak{F} &= \mathfrak{F} \text{ If } a, b \in \mathfrak{F} \text{ then } a \bullet b = (a^\perp b^\perp)^\perp \in (\mathfrak{B}\mathfrak{B})^\perp = \mathfrak{F}. \\ \frac{1}{2}\mathfrak{F} * \frac{1}{2}\mathfrak{F} &= \frac{1}{2}\mathfrak{F} \text{ If } a, b \in \frac{1}{2}\mathfrak{F} \text{ then } 2(a * b) = (2a) \bullet (2b) \in \mathfrak{F} \bullet \mathfrak{F} = \mathfrak{F} \text{ so } a * b \in \frac{1}{2}\mathfrak{F}. \\ \mathfrak{c} \cap -\mathfrak{c} &= \{0\} \text{ If } a^*a \preceq na \text{ and } a^*a \preceq -ma \text{ then } (m+n)a^*a \preceq (mn - mn)a = 0.\end{aligned}$$

$\mathfrak{c} + \mathfrak{c} = \mathfrak{c}$ For all $a, b \in A$, we have $0 \preceq (a^* - b^*)(a - b) = a^*a - a^*b - b^*a + b^*b$ so

$$a^*b + b^*a \preceq a^*a + b^*b.$$

Thus if $a^*a \preceq na$ and $b^*b \preceq mb$ then

$$\begin{aligned} (a^* + b^*)(a + b) &\preceq a^*a + a^*b + b^*a + b^*b \\ &\preceq 2(a^*a + b^*b) \\ &\preceq 2(ma^*a + nb^*b) \\ &\preceq 4mn(a + b). \end{aligned}$$

□

Thus we get preorders $\preceq^{\mathfrak{B}}$ and $\preceq^{\mathfrak{F}}$ and a partial order $\preceq^{\mathfrak{c}}$ defined by

$$\begin{aligned} a \preceq^{\mathfrak{B}} b &\Leftrightarrow a \in \mathfrak{B}b. \\ a \preceq^{\mathfrak{F}} b &\Leftrightarrow b \in a \bullet \mathfrak{F}. \\ a \preceq^{\mathfrak{c}} b &\Leftrightarrow b \in a + \mathfrak{c}. \end{aligned}$$

Noting that $a^* = c^*b^* \Leftrightarrow a = bc \Leftrightarrow a^\perp = b^\perp \bullet c^\perp$, we have

$$a^* \preceq^{\mathfrak{B}} b^* \Leftrightarrow b^\perp \preceq^{\mathfrak{F}} a^\perp.$$

Also $\mathfrak{c} \subseteq \mathfrak{r}$ and, if $a = cb$ and $c \in \mathfrak{B}$, then $a^*a = b^*c^*cb \preceq b^*b$ so

$$(5.1) \quad \preceq^{\mathfrak{c}} \subseteq \preceq^{\mathfrak{r}}.$$

$$(5.2) \quad \preceq^{\mathfrak{B}} \subseteq \preceq^*.$$

Moreover, (5.1) is almost always a strict inclusion, as $\preceq^{\mathfrak{c}} \cap \equiv \text{is } =$, i.e.

$$\mathfrak{c} \cap A_{\text{sk}} = \{0\}.$$

For if $a \in A_{\text{sk}}$ then $a^*a \preceq na \equiv na^*$ implies $2a^*a \preceq n(a + a^*) = 0$ and hence $a = 0$. So if $\mathfrak{c} = \mathfrak{r}$ then $A_{\text{sk}} = \{0\}$, which means $*$ is the identity and hence A is commutative. Even this does not guarantee $\mathfrak{c} = \mathfrak{r}$, for example if $A = \mathbb{Z}^{\mathbb{N}}$ then

$$(1, 4, 9, \dots) \in |A|^2 \setminus \mathfrak{c}.$$

We should also point out here that in C^* -algebras, the various subsets we have defined correspond to their Banach algebra counterparts. Specifically, with $V(a)$ denoting the numerical range of a (see [BD73]), for C^* -algebra A we have

$$\begin{aligned} |A|^2 &= A_\Sigma = A_+ = \{a \in A : V(a) \subseteq \mathbb{R}_+\} \\ &\subseteq \mathbb{R}_+ \mathfrak{F} = \mathfrak{c} \subseteq \mathfrak{r} = \{a \in A : V(a) \subseteq \mathbb{R}_+ + i\mathbb{R}\}. \\ \mathfrak{B} &= \{a \in A : \|a\| \leq 1\}. \\ \preceq^* &\subseteq \preceq^{\mathfrak{r}} \quad \text{on } A_+. \\ \preceq^* &= \preceq^{\mathfrak{B}} \quad \text{if } A \text{ is a von Neumann algebra.} \end{aligned}$$

For the last two results see [Bla13] Proposition II.3.1.10 and [Ped98] Theorem 2.1.

6. ORTHOGONALITY

We can now say more about the orthogonality relation \perp defined §2.

- (6.1) $b^* \perp a^* \Leftrightarrow a \perp b.$
(6.2) $a^*a \perp b \Leftrightarrow a \perp b.$
(6.3) For $a \in \mathfrak{c} \cup A_+$ $b^*a \perp b \Leftrightarrow a \perp b.$
(6.4) For $a \in \mathfrak{c} \cup A_+$ $a \preceq^{\mathfrak{r}} c \perp b \Rightarrow a \perp b.$
(6.5) For $a \in \mathfrak{r}$ $a \preceq^{\mathfrak{c}} c \perp b \Rightarrow a \perp b.$
(6.6) For $a \in \mathfrak{r}$ and $c \in \mathfrak{c} \cup A_+$ $a + c \perp b \Leftrightarrow a \perp b$ and $c \perp b.$
(6.7) For $a \in \mathfrak{c} \cup A_n$ $a^* \perp b \Leftrightarrow a \perp b.$
(6.8) For $a \in \mathfrak{c} \cup A_+$ and $b \in \mathfrak{c} \cup A_n$ $ba \perp bc \Leftrightarrow a \perp bc.$
(6.9) For $a \in \mathfrak{c} \cup A_n$ $a^2 \perp b \Leftrightarrow a \perp b.$

Proof.

- (6.1) If $ab = 0$ then $b^*a^* = (ab)^* = 0.$
(6.2) If $a^*ab = 0$ then $b^*a^*ab = 0$ so $ab = 0$, by (3.1).
(6.3) If $b^*ab = 0$ and $a^*a \preceq na$ then

$$0 \preceq b^*a^*ab \preceq nb^*ab = nb^*0 = 0.$$

so $b^*a^*ab = 0$, by (4.1), thus $ab = 0$, by (3.1). While if $na = c_1^*c_1 + \dots + c_n^*c_n$ then $b^*c_k^*c_kb = 0$, for all k , by (B) and $nb^*ab = 0$. Then (3.1) yields $c_kb = 0$ so $c_k^*c_kb = 0$, for all k . Summing yields $nab = 0$ and hence $ab = 0$.

- (6.4) If $a^*a \preceq na$ then $ab = 0$ follows from (3.1) and (4.1) as

$$0 \preceq b^*a^*ab \preceq nb^*ab \preceq nb^*cb = nb^*0 = 0.$$

If $a \in A_+$ then $0 \preceq b^*ab \preceq b^*cb = 0$ so $a \perp b$, by (4.1) and (6.3).

- (6.5) As $c - a \in \mathfrak{r}$, $(c^* - a^*)(c - a) \preceq n(c - a)$, for some n . Thus

$$b^*(c^* - a^*)(c - a)b \preceq nb^*(c - a)b \preceq nb^*cb = nb^*0 = 0$$

so $(c - a)b = 0$, by (3.1) and (4.1). Again using $cb = 0$, we have $ab = 0$. \square

- (6.6) If $a \perp b$ and $c \perp b$, certainly $a + c \perp b$. The converse is (6.4) and (6.5).
(6.7) If $a \in \mathfrak{c}$ then $a^* \equiv a \perp b$ yields $a^* \perp b$, by (6.4). While if $a \in A_n$, this follows from (3.1) and $(ab)^*(ab) = b^*a^*ab = b^*aa^*b = (a^*b)^*(a^*b).$
(6.8) By (6.7), (3.1) and (6.3), $bab = 0 \Rightarrow b^*abc = 0 \Rightarrow c^*b^*abc = 0 \Rightarrow abc = 0.$
(6.9) If $a \in \mathfrak{c}$ then $a^* \equiv a \perp ab$ so $a^*ab = 0$, by (6.4). If $a \in A_n$ then $b^*a^*aa^*ab = b^*a^*a^2b = 0$ so again $a^*ab = 0$, by (3.1). Now $ab = 0$, by (6.2).

Note that (6.8) can fail for $a \in A_{\text{sa}}$, even when $c = 1$, for example when $A = M_2$, $a = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ we have $ab = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \neq 0 = bab$.

Corollary 1. *Orthogonality is symmetric on $\mathfrak{c} \cup A_n$.*

Proof. For $a, b \in \mathfrak{c} \cup A_n$, we have $a \perp b \Leftrightarrow b \perp a$ by

$$(6.10) \quad \begin{array}{ccccccc} a \perp b & & a \perp b^* & \Leftrightarrow & a^* \perp b^* & & a^* \perp b \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ b^* \perp a^* & \Leftrightarrow & b \perp a^* & & b \perp a & \Leftrightarrow & b^* \perp a, \end{array}$$

using (6.1) for the \Downarrow 's and (6.7) for the \Leftrightarrow 's. \square

Corollary 2. *There are no non-zero nilpotents in $(\mathfrak{c} \cup A_+)(\mathfrak{c} \cup A_+) \cup A_n$.*

Proof. Iterating (6.8) shows that $(ab)^n = 0 \Rightarrow ab = 0$, for all $a, b \in \mathfrak{c} \cup A_+$. By the $b = 1$ case, $a \in A_n$ and $a^n = 0 \Rightarrow (a^*a)^n = a^{*n}a^n = 0 \Rightarrow a^*a = 0 \Rightarrow a = 0$. \square

In fact, iterating (6.8) and (6.9) shows that, for all $a, b \in \mathfrak{c}$ and $l, m, n \in \mathbb{N}$,

$$(6.11) \quad a^l(ba)^m b^n = 0 \Leftrightarrow a \perp b \Leftrightarrow a^l b(ab)^m a^n = 0.$$

As $A_+^2 \subseteq A_{\text{sa}}^2 \subseteq A_+$, this extends to arbitrary products in A_+ , i.e. whenever $c_1, c_2, \dots, c_n \in \{a, b\} \subseteq A_+$,

$$c_1 c_2 \cdots c_n = 0 \Rightarrow a \perp b.$$

So there are no non-zero nilpotents in the $(*)$ -subsemigroup generated by $a, b \in A_+$. This also applies to $|A|^2$ for any proper $*$ -semigroup A (see [Bic15] Corollary 3.6).

Unfortunately, (6.11) does not extend to arbitrary products in \mathfrak{c} . For every $a \in \mathbb{C}$ has a cube-root b with $\arg(b) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, so if $A = \mathbb{C}$ then $A = \mathbb{C}^3$. Thus if $A = M_n$ then $A_n \subseteq \mathfrak{c}^3$, as normal matrices are diagonalizable, by the spectral theorem. Now by the example mentioned before Corollary 1, we have $a, b \in \mathfrak{c}$ with $ba^3b = 0 \neq ab$.

7. FIXATORS

For the fixator relation \ll defined in §2, we immediately see that

$$a \ll b \Leftrightarrow a = ab \Leftrightarrow b = a \bullet b \Leftrightarrow 0 = ab^\perp \Leftrightarrow a \perp b^\perp.$$

Thus the results in §6 for \perp yield corollaries for \ll , e.g. by (6.4), (6.5) and (6.7),

$$(7.1) \quad \text{For } a \in \mathfrak{c} \cup A_+ \quad a \preceq^{\mathfrak{r}} c \ll b \Rightarrow a \ll b.$$

$$\text{For } a \in \mathfrak{r} \quad a \preceq^{\mathfrak{c}} c \ll b \Rightarrow a \ll b.$$

$$(7.2) \quad \text{For } a \in \mathfrak{c} \cup A_n \quad a^* \ll b \Leftrightarrow a \ll b.$$

Together with (6.1) and $a^* \preceq b^* \Leftrightarrow a \preceq b \Leftrightarrow b^\perp \preceq a^\perp$ (for $\preceq^{\mathfrak{c}}$ too), we then have

$$(7.3) \quad \text{For } b \in \mathfrak{c}^\perp \cup A_+^\perp \quad a \ll c \preceq^{\mathfrak{r}} b \Rightarrow a \ll b.$$

$$\text{For } b \in \mathfrak{r}^\perp \quad a \ll c \preceq^{\mathfrak{c}} b \Rightarrow a \ll b.$$

$$(7.4) \quad \text{For } b \in \mathfrak{c}^\perp \cup A_n \quad a \ll b^* \Leftrightarrow a \ll b.$$

And by Corollary 1, for all $a \in \mathfrak{c} \cup A_n$ and $b \in \mathfrak{c}^\perp \cup A_n$, we have

$$a \ll b \Leftrightarrow a \perp b^\perp \Leftrightarrow b^\perp \perp a \Leftrightarrow b^\perp \ll a^\perp.$$

So on $\frac{1}{2}\mathfrak{F}$ and A_n , $a \mapsto a^*$ and $a \mapsto a^\perp$ are \ll -isotone and \ll -antitone bijections.

We can also replace $\preceq^{\mathfrak{r}}$ and $\preceq^{\mathfrak{c}}$ above with \ll , $\preceq^{\mathfrak{B}}$, $\preceq^{\mathfrak{F}}$ or \preceq^* , or even the preorders $\preceq^A \subseteq \ll \cap \preceq^{\mathfrak{B}}$ and $\preceq^\bullet \subseteq \preceq^{\mathfrak{F}}$ defined by

$$a \preceq^A b \Leftrightarrow a \in Ab.$$

$$a \preceq^\bullet b \Leftrightarrow b \in a \bullet A.$$

- (7.5) $a \preceq^A c \ll b \Rightarrow a \ll b.$
 (7.6) For $b \in \mathfrak{B}$ $a \preceq^* c \ll b \Rightarrow a \ll b.$
 (7.7) $a \ll c \preceq^\bullet b \Rightarrow a \ll b.$
 (7.8) For $b \in \mathfrak{B}$ $a \ll c \preceq^{\mathfrak{B}} b \Rightarrow a \ll b^*b.$
 (7.9) For $b \in \frac{1}{2}\mathfrak{F}$ $a \ll c \preceq^{\mathfrak{B}} b \Rightarrow a \ll b.$
 (7.10) For $b \in \frac{1}{2}\mathfrak{F}$ and $c \in \mathfrak{B}$ $a \ll cb \Leftrightarrow a \ll b$ and $a \ll c.$
 (7.11) For $b \in \frac{1}{2}\mathfrak{F}$ $a \ll c \preceq^* b \Rightarrow a \ll b.$
 (7.12) For $a \in \frac{1}{2}\mathfrak{F}$ and $b \in \mathfrak{B}$ $a \preceq^{\mathfrak{F}} c \ll b \Rightarrow a \ll b.$

Proof.

- (7.5) See (2.2).
 (7.6) If $a \preceq^* c \ll b$ then $a^*a \preceq c^*c \preceq^A c \ll b$ so $a^*a \ll b$, by (7.1) and (7.5), as $a^*a \in |\mathfrak{B}|^2 \subseteq \frac{1}{2}\mathfrak{F} \subseteq \mathfrak{c}$. Then $a^*ab^\perp = 0$ gives $ab^\perp = 0$, by (6.1).
 (7.7) Like in (2.2), if $a \ll c \preceq^\bullet b$ then $b = c \bullet d$ so $a \bullet b = a \bullet c \bullet d = c \bullet d = b$.
 (7.8) If $b, c \in \mathfrak{B}$ and $a \ll cb \in \mathfrak{B}\mathfrak{B} = \mathfrak{B}$ then $a \ll b^*c^*$, by (7.4). But $a \ll d, e$ implies $a \ll de$ so $a \ll b^*c^*cb \preceq b^*b$ and hence $a \ll b^*b$, by (7.3).
 (7.9) By (7.8), $a \ll b^*b \preceq b$ so $a \preceq b$, by (7.1).
 (7.10) If $a \ll c, b$ then $acb = ab = a$. While if $a \ll cb$ then $a \ll b$, by (7.9). Then $a \ll (cb)^* = b^*c^*$ and $a \ll b^*$ so $a = ab^*c^* = ac^*$, i.e. $a \ll c^*$ so $a \ll c$.
 (7.11) As $c \preceq^* b \in \frac{1}{2}\mathfrak{F} \subseteq \mathfrak{B}$, we have $c^*c \preceq b^*b \preceq 1$ so $c \in \mathfrak{B}$ too. Thus $a \ll 1c \preceq^* b$ implies $a \ll c^*c \preceq b^*b \preceq b$, by (7.8), so $a \ll b$, by (7.1).
 (7.12) As $\frac{1}{2}\mathfrak{F} = \frac{1}{2}\mathfrak{F}^*$, $a^*a \preceq a \Leftrightarrow aa^* \preceq a^* \Leftrightarrow a \preceq a + a^* - aa^*$ so

$$\frac{1}{2}\mathfrak{F} = \{a \in A : a \preceq a \bullet a^*\}.$$

If $a \preceq^{\mathfrak{F}} c \ll b$ then $a \bullet d \ll b$, for some $d \in \mathfrak{F}$, so $b^\perp \ll a^\perp d^\perp \equiv d^{\perp*} a^{\perp*}$. Thus $b^\perp \ll a^\perp a^{\perp*}$, by (7.8), so $a \preceq a \bullet a^* \ll b$ and $a \ll b$, by (7.1). \square

Corollary 3. \ll is auxiliary to $\preceq^{\mathfrak{r}}, \preceq^{\mathfrak{c}}, \preceq^{\mathfrak{B}}, \preceq^{\mathfrak{F}}$ and \preceq^* on $\frac{1}{2}\mathfrak{F}$.

Proof. By the results above, it only remains to show that $\ll \subseteq \preceq^{\mathfrak{r}}, \preceq^{\mathfrak{c}}, \preceq^{\mathfrak{B}}, \preceq^{\mathfrak{F}}, \preceq^*$ on $\frac{1}{2}\mathfrak{F}$. Actually $\ll \subseteq \preceq^{\mathfrak{B}} (\subseteq \preceq^*$ by (5.2)) is immediate on $\frac{1}{2}\mathfrak{F} \subseteq \mathfrak{B}$, as is $\ll \subseteq \preceq^{\mathfrak{F}}$ on $\frac{1}{2}\mathfrak{F} \subseteq \mathfrak{F}$, remembering that $a \ll b \Leftrightarrow b = a \bullet b$. Lastly, for $\ll \subseteq \preceq^{\mathfrak{r}}, \preceq^{\mathfrak{c}}$ on $\frac{1}{2}\mathfrak{F}$, if $a, b \in \frac{1}{2}\mathfrak{F} \subseteq \mathfrak{c}$ then $a \ll b$ implies $a^* \ll b$, by (7.2), so

$$(b^* - a^*)(b - a) = b^*b - a - a^* + a^*a \preceq b - a - a^* + a^* = b - a.$$

Thus $b - a \in \frac{1}{2}\mathfrak{F} \subseteq \mathfrak{c} \subseteq \mathfrak{r}$. Alternatively, by $a \ll b$ and $\frac{1}{2}\mathfrak{F} * \frac{1}{2}\mathfrak{F} = \frac{1}{2}\mathfrak{F}$,

$$b - a = a + b - 2a = a + b - 2ab = a * b \in \frac{1}{2}\mathfrak{F} \subseteq \mathfrak{c} \subseteq \mathfrak{r}. \quad \square$$

We now examine the \ll -lattice structure of subsets containing $A_+^1 = \frac{1}{2}\mathfrak{F} \cap A_{\text{sa}}$.

- (7.13) $A_+ \subseteq B \subseteq A \Rightarrow B$ is a \ll -semilattice.
 (7.14) $A_+^1 \subseteq B \subseteq \frac{1}{2}\mathfrak{F} \Rightarrow B$ is a \ll -lattice.
 (7.15) $2 \in A^{-1}$ and $A_+^1 \subseteq B \subseteq \mathfrak{B} \Rightarrow B$ is a \ll -lattice.

Proof. Iterating (6.6) and (7.10), we see that sums in $\mathfrak{c} \cup A_+$ are \ll -supremums and products in $\frac{1}{2}\mathfrak{F}$ are \ll -infimums, i.e. (with the product taken in any order)

$$(7.16) \quad \text{For finite } F \subseteq \mathfrak{c} \cup A_+ \quad \sum F = \bigvee F.$$

$$(7.17) \quad \text{For finite } F \subseteq \frac{1}{2}\mathfrak{F} \quad \prod F = \bigwedge F.$$

As $a^*a + b^*b \in A_+$, for all $a, b \in A$, (7.13) follows from (6.2) and (7.16). Likewise, as $a^*b^*ba \in |\mathfrak{B}|^2 \subseteq A_+^1$, for all $a, b \in \frac{1}{2}\mathfrak{F} \subseteq \mathfrak{B}$, and $a \mapsto a^\perp$ is a \ll -antitone bijection, (7.14) follows from (7.4) and (7.17). If $2 \in A^{-1}$ then (7.15) follows from (7.14), $a = \ll a^*a \in \frac{1}{2}\mathfrak{F}$ and $a = \ll \frac{1}{2}(1+a) \in \frac{1}{2}\mathfrak{F}$, for all $a \in \mathfrak{B}$, as

$$a \ll b \Leftrightarrow a \perp b^\perp \Leftrightarrow a \perp \frac{1}{2}b^\perp \Leftrightarrow a \ll (\frac{1}{2}b^\perp)^\perp = \frac{1}{2}(1+b). \quad \square$$

Thus $\ll \circ \ll = \ll \Leftrightarrow \ll_F \circ \ll_F = \ll_F$ on $\frac{1}{2}\mathfrak{F}$ and A_+^1 (and \mathfrak{B} and A_{sa}^1 if $2 \in A^{-1}$). In fact, it does not matter which subset we consider as $a = \ll a^*a = \ll a$, for $a \in \frac{1}{2}\mathfrak{F}$, and, when we identify B with equality on B (i.e. the relation $= \cap B \times B$),

$$\ll \circ A_+^1 \circ \ll = \ll \circ \frac{1}{2}\mathfrak{F} \circ \ll = \ll \circ \mathfrak{B} \circ \ll.$$

Proof. If $a \ll b \ll c$ for $b \in \mathfrak{B}$ then $a \ll b^*b \ll c$, by (6.2) and (7.8). \square

For (possibly non-unital) C*-algebra A , $\perp = \ll \circ \perp$ on A_+^1 is the defining property of a SAW*-algebra (see [Ped86]). As above, we see that A is SAW* iff $\perp = \ll \circ \perp$ on $\frac{1}{2}\mathfrak{F}$ or \mathfrak{B} iff A is ‘Riesz SAW*’ in that $\perp_F = \ll_F \circ \perp_F$ on A_+^1 , $\frac{1}{2}\mathfrak{F}$ or \mathfrak{B} . If A is a unital C*-algebra then $\perp = \ll \circ \perp$ is equivalent to $\ll = \ll \circ \ll$ so

$$A \text{ is SAW*} \Leftrightarrow \ll \text{ has (Riesz) interpolation on } A_+^1, \frac{1}{2}\mathfrak{F} \text{ or } \mathfrak{B}.$$

8. PROJECTIONS

Here we consider the idempotents and projections

$$\mathcal{I} = \{p \in A : p \ll p\}.$$

$$\mathcal{P} = \{p \in A : p \ll p^*\}.$$

Note $\mathcal{P} \subseteq |\mathfrak{B}|^2 \subseteq A_{\text{sa}}$ immediately yields $\mathcal{P} = \mathcal{I} \cap |\mathfrak{B}|^2 \subseteq \mathcal{I} \cap A_{\text{sa}}$, even in an arbitrary *-semigroup. In fact, by [Ber72] §2 Exercise 1A, we have $\mathcal{P} = \mathcal{I} \cap A_n$, even in an arbitrary proper *-ring (see below). Thus \ll is a partial order on \mathcal{P} , as \ll is reflexive on \mathcal{I} and antisymmetric on A_{sa} . Reflexivity combined with auxiliarity on $\mathcal{P} \subseteq \frac{1}{2}\mathfrak{F}$ immediately yields

$$\ll = \preceq^{\mathfrak{r}}, \preceq^*, \preceq^c, \preceq^A, \preceq^\bullet \quad \text{on } \mathcal{P}.$$

$$\text{Moreover } \mathcal{P} = \mathcal{I} \cap (A_n \cup \mathfrak{r} \cup \mathfrak{r}^\perp \cup A_{\text{sa}}A_+^\perp \cup A_+^\perp A_{\text{sa}}),$$

and hence $\mathcal{P} = \mathcal{I} \cap \mathfrak{B} = \mathcal{I} \cap \mathfrak{F} = \mathcal{I} \cap \frac{1}{2}\mathfrak{F}$, as $|\mathfrak{B}|^2 \subseteq \mathfrak{B}$, \mathfrak{F} , $\frac{1}{2}\mathfrak{F} \subseteq \mathfrak{r} \cup \mathfrak{r}^\perp$.

Proof.

($A_n \cap \mathcal{I} = \mathcal{P}$) If $p \in A_n$ then $p \ll p$ implies $p \ll p^*$, by (7.4).

($\mathfrak{r} \cap \mathcal{I} = \mathcal{P}$) If $p \in \mathfrak{r}$ then $p + p^* \in A_+$. If $p \in \mathcal{I}$ too then

$$p^\perp(p + p^*)p^{\perp*} = p^\perp pp^{\perp*} + p^\perp p^* p^{\perp*} = 0p^{\perp*} + p^\perp 0 = 0.$$

By (6.3), $pp^{\perp*} = (p + p^*)p^{\perp*} = 0$ so $p = pp^*$.

($\mathfrak{r}^\perp \cap \mathcal{I} = \mathcal{P}$) If $a \in \mathcal{I}$ then $a^\perp a^\perp = 1 - a - a + a = a^\perp$, so $\mathcal{I} = \mathcal{I}^\perp$. Thus $\mathcal{P} = \mathcal{I} \cap A_n = \mathcal{I}^\perp \cap A_n^\perp = \mathcal{P}^\perp$ and hence $\mathcal{P} = \mathcal{P}^\perp = \mathcal{I}^\perp \cap \mathfrak{r}^\perp = \mathcal{I} \cap \mathfrak{r}^\perp$.

$(A_{\text{sa}}A_+^\perp \cap \mathcal{I} = \mathcal{P})$ If $a \in A_{\text{sa}}, b \in A_+^\perp$ and $ab = abab$ then $bab^\perp ab = babab - bab = bab - bab = 0$ and hence $bab^\perp = 0$, by (6.3), so $ba = bab = (bab)^* = ab \in A_{\text{sa}} \cap \mathcal{I} = \mathcal{P}$.
 $(A_+^\perp A_{\text{sa}} \cap \mathcal{I} = \mathcal{P})$ Note $\mathcal{P} = \mathcal{P}^* = (A_{\text{sa}}A_+^\perp \cap \mathcal{I})^* = A_+^\perp A_{\text{sa}} \cap \mathcal{I}$. \square

Another fact possibly worth noting is the following.

$$(8.1) \quad \text{For } p \in \mathcal{P} \text{ and } a \in A_+, \quad p \ll a \Rightarrow p \preceq a.$$

Proof. If $p = pa$ then $p = ap$ so $a - p = a - ap = ap^\perp = ap^{\perp 2} = p^\perp ap^\perp \in A_+$. \square

We can also use \mathcal{I} to characterize \ll, \perp and commutativity on \mathcal{P} as follows.

$$(8.2) \quad pq \in \mathcal{I} \Leftrightarrow pq = qp.$$

$$(8.3) \quad p + q \in \mathcal{I} \Leftrightarrow p \perp q.$$

$$(8.4) \quad p - q \in \mathcal{I} \Leftrightarrow q \ll p.$$

Proof. The \Leftarrow parts are immediate, even in an arbitrary *-ring.

(8.2) As $A_{\text{sa}}A_+^\perp \cap \mathcal{I} = \mathcal{P}$, certainly $\mathcal{P}\mathcal{P} \cap \mathcal{I} \subseteq A_{\text{sa}}$ so $pq = (pq)^* = qp$.

(8.3) If $p + q = (p + q)^2 = p + pq + qp + q$ then $pq = -qp$ so $pq = ppq = -pqp \in A_{\text{sa}}$ and hence $pq = (pq)^* = qp = -pq$ which, as A is torsion-free, means $pq = 0$.

(8.4) If $p - q = (p - q)^2 = p - pq - qp + q$ then $2q = pq + qp$ so $2pq = pq + pqp$ and hence $pq = pqp = (pqp)^* = qp$. Thus $2q = 2qp$ so again $q = qp$. \square

9. PRODUCTS

In this section we make the following additional standing assumption.

(C) $A_+A_+ \cap A_{\text{sa}} = A_+$.

As $A_+ = \frac{1}{n}A_+$, the apparently weaker assumption $A_\Sigma A_\Sigma \cap A_{\text{sa}} \subseteq A_+$ would actually suffice. Also, if $a, b \in A_{\text{sa}}$ then $ab \in A_{\text{sa}} \Leftrightarrow ab = (ab)^* = ba$, so (C) is just saying that products of commuting *-positive elements are *-positive (which holds for C*-algebra A – see [KR97] Theorem 4.2.2(iv)). Using (C), we have the following.

$$(9.1) \quad A_+^\perp = A_+ \cap \mathfrak{B} = A_+ \cap \mathfrak{r}^\perp.$$

$$(9.2) \quad A_+A_+ \cap -A_+ = \{0\}.$$

$$(9.3) \quad \text{For } a \in A_+ \quad ab + ba = 0 \Rightarrow a \perp b.$$

Proof.

(9.1) Note $A_{\text{sa}} \cap \frac{1}{2}\mathfrak{F} \subseteq A_{\text{sa}} \cap \mathfrak{r} \cap \mathfrak{B} = A_+ \cap \mathfrak{B} = A_+ \cap \mathfrak{F}^\perp \subseteq A_+ \cap \mathfrak{r}^\perp$. If $a \in A_+ \cap \mathfrak{r}^\perp$ then $a, a^\perp \in A_+$ so $aa^\perp = a^\perp a \in A_+$, by (C), and hence $a^2 \preceq a$.

(9.2) Combine (4.1) and (C).

(9.3) If $ab = -ba$ then $abb^* = -bab^* \in -A_+$ so $a \perp b$, by (6.2) and (9.2). \square

Actually, from now on, all we need is the strengthening of (4.1) given in (9.2).

We call $a, b \in A_+$ with $a \perp b$ a *decomposition* of $c \in A_{\text{sa}}$ if $c = a - b$. The following generalizes a standard result for C*-algebras (see [KR97] Proposition 4.2.3(iii)).

Theorem 1. *Decompositions are unique.*

Proof. If $a - b = c - d$ and $ab = 0 = cd$, for some $a, b, c, d \in A_+$, then

$$(a - c)^2 b = (b - d)(a - c)b = -(b - d)cb = -bcb$$

so $b \perp c$, by (6.3) and (9.2). Likewise, $a \perp d$ so

$$a^2 = a(a - b) = a(c - d) = ac = (a - b)c = (c - d)c = c^2$$

Thus $(a - c)^2 = a^2 - ac - ca + c^2 = 0$ so $a = c$, by (3.1), and hence $b = d$. \square

For $B \subseteq A$ let $B' = \{a \in A : \forall b \in S(ab = ba)\}$. Another standard C^* -algebra fact is that any $a \in A_+$ has a $*$ -positive square-root in $C^*(a) \subseteq \{a\}''$, where $C^*(a)$ is the C^* -subalgebra generated by A . This generalizes too as follows which, for example, implies that (8.1) extends to $p \in A_+^{\perp}$.

Theorem 2. *For $a \in A_+$ we have $a \in \{a^2\}''$.*

Proof. If $ba^2 = a^2b$ then $a^2b^* = b^*a^2$ and

$$a(ab - ba) = a^2b - aba = ba^2 - aba = (ba - ab)a = -(ab - ba)a.$$

By (9.3), $a \perp ab - ba$ so $a^2b = aba$ and $b^*a^2 = ab^*a$. Also $bb^*a^2 = ba^2b^* = a^2bb^*$ so the same argument applied to bb^* instead of b yields $a^2bb^* = abb^*a$. Thus

$$(ab - ba)(b^*a - ab^*) = abb^*a - abab^* - bab^*a + ba^2b^* = a^2bb^* - a^2bb^* - bb^*a^2 + bb^*a^2 = 0.$$

Thus $ab = ba$, by (3.1). As $b \in \{a^2\}'$ was arbitrary, $a \in \{a^2\}''$. \square

By Theorem 2, the positive square-root axiom (PSR) given in [Ber72] §13 Definition 9 reduces to $A_+ = A_+^2$ in the presence of (A) and (9.2). These positive square-roots are even unique, by [Ber72] §13 Exercise 10. Indeed, if $a, b \in A_+$, $a^2 = b^2$ and $ab = ba$ then $(a + b)(a - b) = a^2 - ab + ba - b^2 = 0$ so

$$0 = -(a - b)(a + b)(a - b) \preceq (a - b)(a - b)(a - b) \preceq (a - b)(a + b)(a - b) = 0.$$

Thus $0 = (a - b)^3 = (a - b)^4$, by (B), and hence $a = b$, by (3.1). Actually, we already have a weak form of (PSR), as (9.1) means $a^2 \preceq a$, for all $a \in A_+ \cap \mathfrak{B}$, so A_+^2 is \preceq -coinitial in $A_+ \cap \mathfrak{B} \setminus \{0\}$.

If $A_+ = A_+^2$, define $|a| = \sqrt{a^*a}$. If $a \in A_{\text{sa}}$ then $|a|^2 = a^*a = a^2$ and hence $|a|a = a|a|$, by Theorem 2, so $(|a| + a)(|a| - a) = |a|^2 - |a|a + a|a| - a^2 = 0$, i.e.

$$(9.4) \quad |a| + a \perp |a| - a.$$

So if $a, -a \preceq |a|$ and 2 is invertible in A then $\frac{1}{2}(|a| + a)$ and $\frac{1}{2}(|a| - a)$ form a decomposition of a . Also (9.4) allows us to extend (9.2) if $\preceq^* \subseteq \preceq^r$ on A_+ , yielding an elementary result which might be new even for C^* -algebra A .

Theorem 3. *If $\preceq^* \subseteq \preceq^r$ on $A_+ = A_+^2$ then $A_+A_+ \cap -\mathfrak{r} = \{0\}$.*

Proof. If $a, b \in A_+$ and $ab \in -\mathfrak{r}$ then $ab \equiv ba \preceq 0$ so

$$(a + b)^2 = a^2 + ab + ba + b^2 \preceq a^2 - ab - ba + b^2 = (a - b)^2.$$

As $\preceq^* \subseteq \preceq^r$ on $A_+ = A_+^2$, we have $a + b \preceq |a - b|$ and hence $2a \preceq |a - b| + a - b$ and $2b \preceq |a - b| + b - a$. By (6.4) and (9.4), $2a \perp 2b$ and hence $a \perp b$. \square

10. BLACKADAR $*$ -RINGS

Throughout this section we merely assume

A is a (possibly non-unital) $*$ -ring.

We define *Blackadar* and, for any $R \subseteq A \times A$, *R-Blackadar* as follows.

$$(10.1) \quad A \text{ is } R\text{-Blackadar} \Leftrightarrow \mathcal{P} \setminus \{0\} \text{ is } R\text{-coinitial in } A \setminus \{0\}.$$

$$(10.2) \quad A \text{ is Blackadar} \Leftrightarrow \forall a \in A \setminus \{0\} \quad \exists p \in \mathcal{P} \setminus \{1\} \quad (\perp a) \subseteq (\ll p).$$

$$\text{So } A \text{ is } \subseteq_{\perp}\text{-Blackadar} \Leftrightarrow \forall a \in A \setminus \{0\} \quad \exists p \in \mathcal{P} \setminus \{0\} \quad (\perp a) \subseteq (\perp p)$$

$$\text{and } A \text{ is } \ll\text{-Blackadar} \Leftrightarrow (\ll a) \neq \{0\} \Rightarrow \exists p \in \mathcal{P} \setminus \{0\} \quad (\ll p) \subseteq (\ll a).$$

Replacing \subseteq with $=$ in (10.2) would define a Rickart *-ring (see [Ber72] §3). Also

$$A \text{ is weakly Rickart} \Leftrightarrow A \text{ is } \subseteq_{\perp}\text{-Blackadar},$$

by [Ber72] §5 Proposition 3 and the following.

Proposition 1. *Every \subseteq_{\perp} -Blackadar *-ring is proper.*

Proof. If $a \neq 0$ and $aa^* = 0$ then we have $p \in \mathcal{P} \setminus \{0\}$ with $p \subseteq_{\perp} a$ so $pa^* = 0 = ap$ and hence $p = pp = 0$, a contradiction. \square

A unital *-ring A is Blackadar iff A is \subseteq_{\perp} -Blackadar, as

$$(\perp a) \subseteq (\ll p \neq 1) \Leftrightarrow (\perp a) \subseteq (\perp p^{\perp} \neq 0).$$

In fact, most Blackadar *-rings are automatically unital, as the following generalization of [Ber72] §3 Proposition 2 shows.

Proposition 2. *Every Blackadar *-ring A is proper and*

$$(10.3) \quad 0 \in (0 \neq)(0 \neq) \Rightarrow \mathcal{P} \neq \{0\} \Leftrightarrow 0 \neq 1 \in A.$$

Proof. First we show (10.3) holds, even under the weaker assumption

$$(10.4) \quad \forall a \in A \setminus \{0\} \exists p \in \mathcal{P} (\perp a) \subseteq (\ll p).$$

For $0 \in (0 \neq)(0 \neq)$ means we have $a, b \neq 0 = ab$. Thus we have $p \in \mathcal{P}$ with $(\perp b) \subseteq (\ll p)$ so $a \ll p$. If $p = 0$ then $a = ap = 0$, a contradiction, which proves

$$0 \in (0 \neq)(0 \neq) \Rightarrow \mathcal{P} \neq \{0\}.$$

Now if $0 \neq p \in \mathcal{P}$ then we have $q \in \mathcal{P}$ with $(\perp p) \subseteq (\ll q)$. Then, for all $a \in A$, $a = ap + ap^{\perp}$ (we interpret ap^{\perp} here as shorthand for $a - ap$) and $ap^{\perp} \perp p$ so $ap^{\perp} \ll q$ and hence $a = ap + ap^{\perp}q = a(p + q - pq)$, i.e. $p + q - pq$ is a right unit for A . In particular, $q = qp + q - qpq$ so $qp = qpq = (qpq)^* = pq$, so $p + q - pq$ is self-adjoint and hence a left unit for A as well, which proves

$$\mathcal{P} \neq \{0\} \Rightarrow 0 \neq 1 \in A.$$

The converse is immediate, and in fact this argument shows that

$$(10.4) \Leftrightarrow 0 \notin (0 \neq)(0 \neq) \text{ or } 1 \in A.$$

If $0 \notin (0 \neq)(0 \neq)$ then A is certainly proper. Otherwise A is unital so A is \subseteq_{\perp} -Blackadar and hence proper, by Proposition 1. \square

If A is unital then $a \ll b \Leftrightarrow a \perp b^{\perp}$ immediately yields $\subseteq_{\ll} = \subseteq_{\perp}$. In the non-unital case we still have the following.

$$(10.5) \quad (\subseteq_{\ll} a) \subseteq (\subseteq_{\perp} a) \quad \text{if } (a \ll) \neq \emptyset.$$

$$(10.6) \quad \subseteq_{\perp} \subseteq \subseteq_{\ll} \quad \text{if } A \text{ is proper.}$$

Proof.

(10.5) If $b \subseteq_{\ll} a \ll c$ then $b \ll c$. If $a \perp d$ too then $a \ll cd^{\perp}$ so $b \ll cd^{\perp}$. Thus $b = bcd^{\perp} = bd^{\perp}$ so $b \perp d$ and hence $(a \perp) \subseteq (b \perp)$, i.e. $b \subseteq_{\perp} a$.

(10.6) If $a \subseteq_{\perp} b \ll c$ then $bc^{\perp}c^{\perp*}a = 0$ so $ac^{\perp}c^{\perp*}a = 0$ and hence $ac^{\perp} = 0$, by properness. Thus $a \ll c$ and hence $(b \ll) \subseteq (a \ll)$, i.e. $a \subseteq_{\ll} b$. \square

$(0 \neq \circ \preceq^A) = (0 \neq)$ and \preceq^A -Blackadar $\Rightarrow \subseteq_{\perp}$ -Blackadar $\Rightarrow \ll$ -Blackadar,
with equivalence holding if

$$(10.7) \quad (0 \neq \circ \ll \circ \subseteq_{\perp}) = (0 \neq).$$

Proof. If $(0 \neq \circ \preceq^A) = (0 \neq)$ then, for any $a \neq 0$, we have $b \preceq^A a$, for some $b \in A \setminus \{0\}$. If A is \preceq^A -Blackadar then $p \preceq^A a$, for some $p \in \mathcal{P} \setminus \{0\}$ so $p \subseteq_{\perp} a$, by (1.4) and (2.3), so A is \subseteq_{\perp} -Blackadar.

If A is \subseteq_{\perp} -Blackadar and $0 \neq b \ll a$ then we have $p \in \mathcal{P} \setminus \{0\}$ with $p \subseteq_{\perp} b \ll a$ so $p \ll a$, by Proposition 1 and (10.6), so A is \ll -Blackadar.

If A is \ll -Blackadar and (10.7) holds then, for all $a \neq 0$, we have $b, c \in A$ with $c \ll b \preceq^A a$ so $p \ll b$, for some $p \in \mathcal{P} \setminus \{0\}$. Thus $p \preceq^A b \preceq^A a$ so $p \preceq^A a$ and hence A is \preceq^A -Blackadar. \square

In a topological semigroup, define a topological version of the Green relation by

$$a \preceq^{\perp} b \Leftrightarrow a \in \overline{Ab}.$$

Corollary 4. For C^* -algebra A ,

$$\preceq^A\text{-Blackadar} \Leftrightarrow \preceq^{\perp}\text{-Blackadar} \Leftrightarrow \subseteq_{\perp}\text{-Blackadar} \Leftrightarrow \ll\text{-Blackadar}.$$

Proof. As multiplication is continuous, $\preceq^A \subseteq \preceq^{\perp} \subseteq \subseteq_{\perp}$. Thus it suffices to show that A satisfies (10.7), which follows from the continuous functional calculus. Specifically, for any $a \in A \setminus \{0\}$, take continuous functions f and g on \mathbb{R} such that $f \ll g$ and $f(|a|^2) \neq 0 = g(x)$, for all x in a neighbourhood of 0, so

$$0 \neq f(a^*a) \ll g(a^*a) \preceq^A a. \quad \square$$

In C^* -algebras, closed left ideals I correspond precisely to hereditary C^* -subalgebras $I \cap I^*$. So A is \preceq^{\perp} -Blackadar iff every hereditary C^* -subalgebra contains a non-zero projection, which is property (SP) from [Bla94]. Thus, for C^* -algebra A ,

$$(10.8) \quad A \text{ is } \subseteq_{\perp}\text{-Blackadar} \Leftrightarrow A \text{ has property (SP)}.$$

Incidentally, for C^* -algebra A we also have $\preceq^{\perp} = \subseteq_{\perp}^*$, where \perp^* is defined on $A \times A^*$ (here A^* is the dual of A) by $a \perp^* \phi \Leftrightarrow \phi[Aa] = \{0\}$ (see [Eff63]).

11. LATTICE STRUCTURE

Throughout this section we assume

$$A \text{ is a } \subseteq_{\perp}\text{-Blackadar } ^*\text{-ring}.$$

Unlike weakly Rickart * -rings, the projections in a \subseteq_{\perp} -Blackadar * -ring may not form a lattice. However, we can still examine supremums and infimums in \mathcal{P} when they do exist, generalizing the weakly Rickart * -ring theory.

First we need the following elementary facts.

$$(11.1) \quad (\ll a) \cap (\ll b) \subseteq (\perp ab^{\perp}).$$

$$(11.2) \quad (a \ll) \cap (b \ll) \subseteq (ab^{\perp} \ll).$$

Proof.

$$(11.1) \quad \text{If } c \ll a, b \text{ then } cab^{\perp} = cb^{\perp} = 0.$$

$$(11.2) \quad \text{If } a, b \ll c \text{ then } ab^{\perp}c = ac - abc = a - ab = ab^{\perp}. \quad \square$$

The following results say \mathcal{P} a complete sublattice of A , in an appropriate sense.

Proposition 3. *Minimal upper bounds in \mathcal{P} are \ll -supremums in A .*

Proof. Say $Q \subseteq \mathcal{P}$ and $Q \ll p \in \mathcal{P}$. If $p \neq \bigvee Q$ in A then $Q \ll a$ but $p \not\ll a$, for some $a \in A$. So $0 \neq r \subseteq_{\perp} a^{\perp} p \ll p$, for some $r \in \mathcal{P}$. Thus $r \ll p$, by (10.6), but

$$Q \subseteq (\ll p) \cap (\ll a) \subseteq (\perp pa^{\perp}) \subseteq (\perp r).$$

So $Q \ll p - r \ll p$ even though $p \neq p - r \in \mathcal{P}$, i.e. p is not minimal. \square

Proposition 4. *Maximal lower bounds in \mathcal{P} are \ll -infimums in A .*

Proof. Say $p \ll Q$ but $p \neq \bigwedge Q$ in A so $a \ll Q$ but $a \not\ll p$, for some $a \in A$. So $0 \neq r \subseteq_{\perp} ap^{\perp} \perp p$, for some $r \in \mathcal{P}$, and hence $r \perp p$. But

$$Q \subseteq (p \ll) \cap (a \ll) \subseteq (ap^{\perp} \ll) \subseteq (r \ll)$$

so $p \ll p + r \ll Q$, even though $p \neq p + r \in \mathcal{P}$, i.e. p is not maximal. \square

For $p, q, r \in \mathcal{P}$ we define

$$r = p^{\perp} \wedge q \iff p \perp r \ll q \text{ and } \{s \in \mathcal{P} : p \perp s \ll q\} \subseteq (\ll r).$$

If A is unital, this coincides the definition of $p^{\perp} \wedge q$ in (1.6). In general, we can still characterize $p^{\perp} \wedge q$ as follows (note $X = Y$ for partially defined expressions X and Y means X is defined iff Y is defined, in which case they coincide).

$$(11.3) \quad p^{\perp} \wedge q = \bigvee_{p \perp s \ll q} s.$$

Proof. If $r = p^{\perp} \wedge q$ then $\bigcap_{p \perp s \ll q} (s \ll) \subseteq (r \ll)$, as $p \perp r \ll q$, and $(r \ll) \subseteq \bigcap_{p \perp s \ll q} (s \ll)$, as $s \ll r$ whenever $p \perp s \ll q$, so $r = \bigvee_{p \perp s \ll q} s$.

Conversely, if $r = \bigvee_{p \perp s \ll q} s$ then $\{s \in \mathcal{P} : p \perp s \ll q\} \subseteq (\ll p^{\perp} r)$ so $r \ll p^{\perp} r$, by Proposition 3. Thus $rpr = 0$ and hence $p \perp r$, by Proposition 1, so $r = p^{\perp} \wedge q$. \square

Incidentally, for C^* -algebra A , (11.3) applies even if A is not \subseteq_{\perp} -Blackadar. Indeed, if $r = \bigvee_{p \perp s \ll q} s$ commutes with p then $p^{\perp} r$ is a projection so the last part still applies even without recourse to Proposition 3. While if r does not commute with p then $\sigma(pr) \neq \{0, 1\}$ so we can apply the continuous functional calculus as in [Bic13] to obtain a projection $t \in C^*(r, p)$ with $r \not\ll t$ and $(\ll p^{\perp} t) = (\ll p^{\perp} r)$ so $t \in \bigcap_{p \perp s \ll q} (s \ll) \setminus (r \ll)$, contradicting $r = \bigvee_{p \perp s \ll q} s$.

For $R \subseteq A \times A$ and $a \in A$, if we define $p = [a]_R \iff a =_R p \in \mathcal{P}$ then

$$[a]_{\ll} = [a]_{\perp}.$$

Proof. If $p = [a]_{\ll}$ then $a =_{\ll} p \ll p$ and hence $a =_{\perp} p$, by (10.5), i.e. $p = [a]_{\perp}$. While if $a =_{\perp} p$ then $a =_{\ll} p$, by Proposition 1 and (10.6). \square

Let $[a] = [a]_{\ll} = [a]_{\perp}$, which is the *right support projection* of a (see [Ber72] §3 Definition 4). Also let $(p \vee q^{\perp}) \wedge q = (q^{\perp} \wedge p)^{\perp} \wedge q$, which is the *Sasaki projection* of p onto q (see [Kal83] §7). By (11.4), this coincides with the right support projection of pq , while (11.5) and (11.6) generalize [Ber72] §5 Proposition 7.

$$(11.4) \quad (p \vee q^{\perp}) \wedge q = [pq].$$

$$(11.5) \quad p \wedge q = [p^{\perp} q]^{\perp} q.$$

$$(11.6) \quad p \vee q = [pq^{\perp}] + q.$$

Proof.

(11.4) If $[pq]$ is defined then $[pq] \ll q$, as $pq \ll q$, so

$$s = [pq]^\perp q = q[pq]^\perp = q - [pq] \in \mathcal{P}.$$

As $pq \ll [pq]$, $ps = pq[pq]^\perp = 0$ so $p \perp s \ll q$. While if $p \perp r \ll q$, for some $r \in \mathcal{P}$, then $pqr = pr = 0$ so $[pq]r = 0$ and hence $rs = r[pq]^\perp = r$, i.e. $r \ll s$. Thus $s = p^\perp \wedge q$ so $(p \vee q^\perp) \wedge q = q - s = [pq]$.

On the other hand, if $p^\perp \wedge q$ is defined then so is

$$s = (p \vee q^\perp) \wedge q = q - (p^\perp \wedge q) = q(p^\perp \wedge q)^\perp = (p^\perp \wedge q)^\perp q.$$

Note $pqs = pq - p(p^\perp \wedge q) = pq$, i.e. $pq \ll s$. If $s \not\subseteq_\perp pq$ then $pq \perp a$ and $sa \neq 0$, for some $a \in A$. Thus $0 \neq r \subseteq_\perp a^*s$, for some $r \in \mathcal{P}$. Then $r \ll s \ll q$ so $pr = pqr = 0$, as $pqsa = pqa = 0$, i.e. $p \perp r$. Thus $r \ll (p^\perp \wedge q)$ so $r = rs = r(q - (p^\perp \wedge q)) = r - r = 0$, a contradiction. So $s = [pq]$, i.e. $(p \vee q^\perp) \wedge q = [pq]$.

(11.5) Note $p^\perp \wedge q = q - ((p \vee q^\perp) \wedge q) =$ and $[pq]^\perp q = q - [pq]$, so all we really need to do is exchange p and p^\perp in the proof of (11.4) above.

(11.6) If $[pq^\perp]$ is defined then $[pq^\perp] \perp q$, as $pq^\perp \perp q$, so $q \ll s = [pq^\perp] + q \in \mathcal{P}$. Now $pq[pq^\perp] = p0 = 0$ so $pqs = pq = pq$ and $pq^\perp q = p0 = 0$ so $pq^\perp s = pq^\perp[pq^\perp] = pq^\perp$. Thus $ps = pqs + pq^\perp s = pq + pq^\perp = p$, i.e. $p \ll s$. While if $p, q \ll r$ then $pq^\perp r = prq^\perp = pq^\perp$ so $[pq^\perp] \ll r$ and thus $s = [pq^\perp] + q \ll r$. Thus $s = p \vee q$.

If $p \vee q$ is defined let $s = (p \vee q) - q = (p \vee q)q^\perp = q^\perp(p \vee q) \in \mathcal{P}$. Then $pq^\perp s = pq^\perp(p \vee q) = p(p \vee q)q^\perp = pq^\perp$, i.e. $pq^\perp \ll s$. If $s \not\subseteq_\perp pq^\perp$ then $pq^\perp a = 0 \neq sa$, for some $a \in A$. Thus $0 \neq r \subseteq_\perp a^*s$, for some $r \in \mathcal{P}$. Then $r \ll s \perp q$ so $pr = pq^\perp r = 0$, as $pq^\perp sa = pq^\perp a = 0$, i.e. $p \perp r$. Thus $p, q \ll r^\perp(p \vee q)$ and hence $p \vee q \ll r^\perp(p \vee q)$, by Proposition 3. Hence $(p \vee q)r(p \vee q) = 0$ so $p \vee q \perp r$, by Proposition 1. But then $r = rs = r(p \vee q)q^\perp = 0$, a contradiction. Thus $s = [pq^\perp]$ so $p \vee q = s + q = [pq^\perp] + q$. \square

Let \top_{\ll} and \top_\perp denote the \ll -incompatibility and \subseteq_\perp -incompatibility relations.

$$(11.7) \quad \text{For } p \in \mathcal{P} \quad a \not\subseteq_\ll p \ll a \Rightarrow \exists q \in \mathcal{P} \setminus \{0\} (p \perp q \ll a).$$

$$(11.8) \quad \text{For } p \in \mathcal{P} \quad p \not\subseteq_\ll a \ll p \Rightarrow \exists q \in \mathcal{P} \setminus \{0\} (a \perp q \ll p).$$

$$(11.9) \quad \text{For } p \in \mathcal{P} \quad p \not\ll a \Rightarrow \exists q \in \mathcal{P} \setminus \{0\} (a \top_{\ll} q \ll p),$$

$$(11.10) \quad \text{For } p \in \mathcal{P} \quad p \not\subseteq_\perp a \Rightarrow \exists q \in \mathcal{P} \setminus \{0\} (a \top_\perp q \ll p),$$

Proof.

(11.7) If $a \not\subseteq_\ll p \ll a$ then we have $b \ll a$ with $b \not\ll p$ and hence $bp^\perp \neq 0$. Thus we have a non-zero projection $q \subseteq_\perp bp^\perp$. As $bp^\perp p = 0$, we have $q \perp p$ and, as $bp^\perp a = ba - bpa = b - bp = bp^\perp$, (10.6) yields $q \ll a$.

(11.8) If $p \not\subseteq_\ll a \ll p$ then we have $b \gg a$ with $p \not\ll b$ and hence $pb^\perp \neq 0$. Thus we have a non-zero projection $q \subseteq_\perp b^{\perp\perp}p$ and hence $q \ll p$, by (10.6). As $a \ll p, b$, we have $b^{\perp\perp}pa^* = b^{\perp\perp}a^* = 0$ and hence $qa^* = 0 = aq$.

(11.9) If $p \not\ll a$, we have a non-zero projection $q \subseteq_\perp a^{\perp\perp}p$. By (10.6), $q \ll p$ so if $r \ll a, q$ then $a^{\perp\perp}pr = a^{\perp\perp}r = 0$ and hence $r = qr = 0$, i.e. $a \top_{\ll} q$.

(11.10) If $p \not\subseteq_{\perp} a$ then $a \perp b$ and $p \not\perp b$, for some $b \in A$. Thus we have a non-zero projection $q \subseteq_{\perp} b^*p$. By (10.6), $q \ll p$ so if $r \subseteq_{\perp} a, q$ then $r \perp b$ and $r \ll p$, by (10.6), so $b^*pr = b^*r = 0$ and hence $r = qr = 0$, i.e. $a \top_{\perp} q$. \square

There are C^* -algebras where (11.7) and (11.8) fail. For example, considering $C([0, 1], M_2)$, every projection $p \neq 0, 1$ has rank 1 everywhere on $[0, 1]$ and hence the required $q \in \mathcal{P}$ does not exist for $a \neq 1$ with $a \not\subseteq_{\perp} p \ll a$ in (11.7), or for $a \neq 0$ with $p \not\subseteq_{\perp} a \ll p$ in (11.8).

If we restrict to $a \in \mathcal{P}$ then (11.7) and (11.8) are just saying that \mathcal{P} is orthomodular, which is immediate (take $q = a - p$ or $p - a$). On the other hand, there are C^* -algebras where A is not orthomodular (w.r.t. \subseteq_{\perp}), e.g. $C([0, 1], \mathbb{K})$, where \mathbb{K} denotes the compact operators on a separable infinite dimensional Hilbert space – see [AB15] Example 4.

Taking $a \in \mathcal{P}$ in (11.9) or (11.10) generalizes [Bic12] Theorem 4.4 as follows.

Corollary 5. *Separativity holds on \mathcal{P} .*

There are C^* -algebras where separativity does not hold on \mathcal{P} . For example, consider $C(X, M_2)$ where $X = \{-1/n : n \in \mathbb{N}\} \cup [0, 1]$, and take everywhere rank 1 projections p and q that coincide on $\{-1/n : n \in \mathbb{N}\}$ but differ on $(0, 1]$. Then $q \top_{\perp} r \ll p$ implies $r = 0$ on $\{-1/n : n \in \mathbb{N}\}$ and hence on $[0, 1]$, by continuity.

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